B. What is a wave?

Wave = a propagating displacement-of-particles-from-their-equilibrium-positions.

There are three main kinds:

Transverse = particles displace about equilibrium position perpendicular to wave velocity
e.g. string waves, certain seismic waves, EM waves, gravity waves, ‘the’ wave …

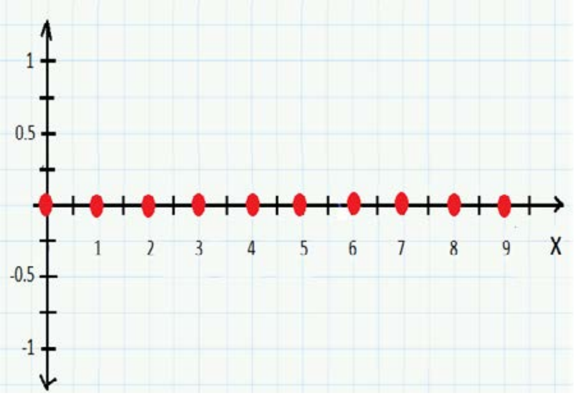
Longitudinal = particles displace about equilibrium position parallel to wave velocity
e.g. sound waves, and other certain seismic waves.

Elliptical = particles displace in circles about equilibrium position.
e.g. water waves.

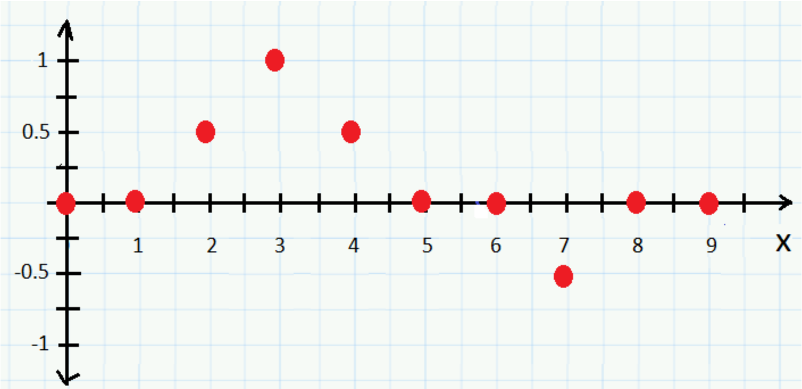
In what follows we’ll want to ascertain how we can graphically/mathematically describe an initial wave pulse, how it will propagate through the medium, the motion the wave will induce in the particles in the medium, the energy the wave carries, etc.

B.1 Description of wave (1D)

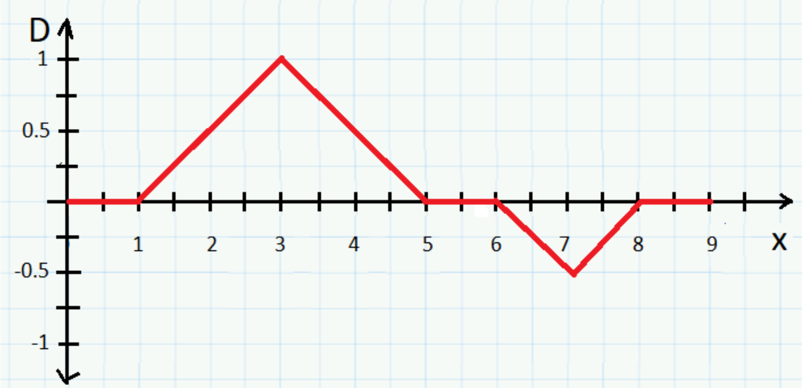
Equilibrium Position



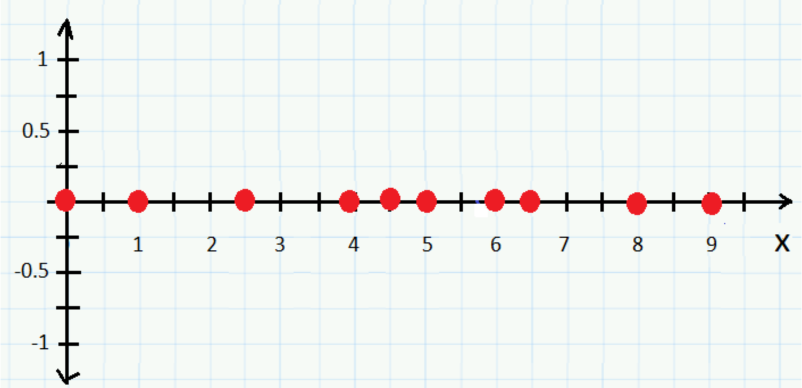
Transverse displacement



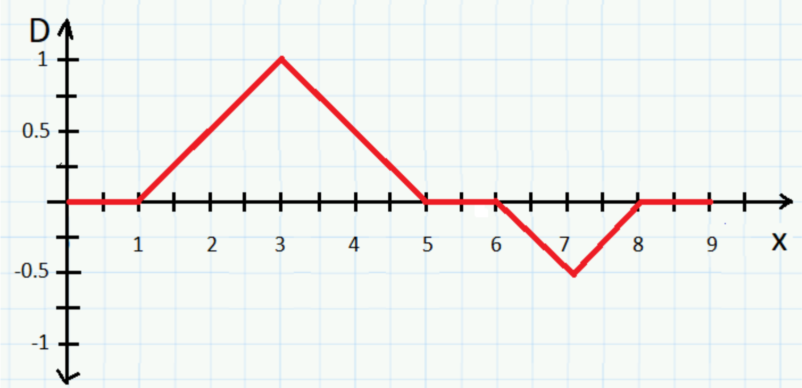
Graphical/Mathematical description



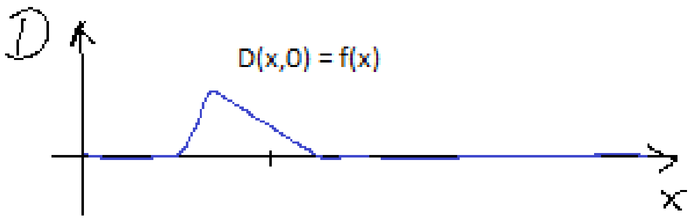
Longitudinal displacement

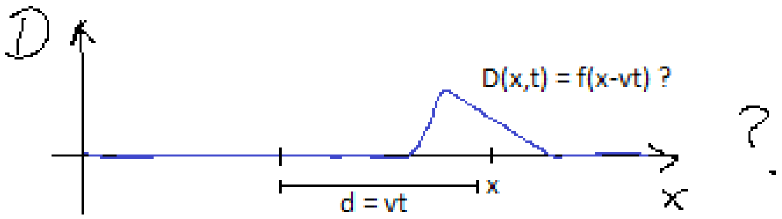


Graphical/Mathematical description



B.1 How does displacement propagate through medium (1D)?





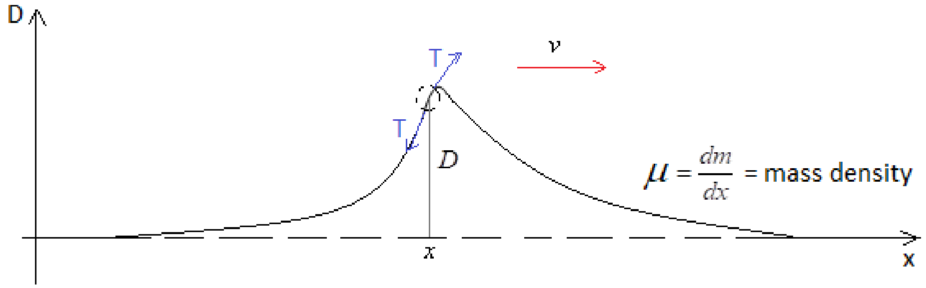
Suppose we create an initial disturbance, of shape D(x,0) = f(x).

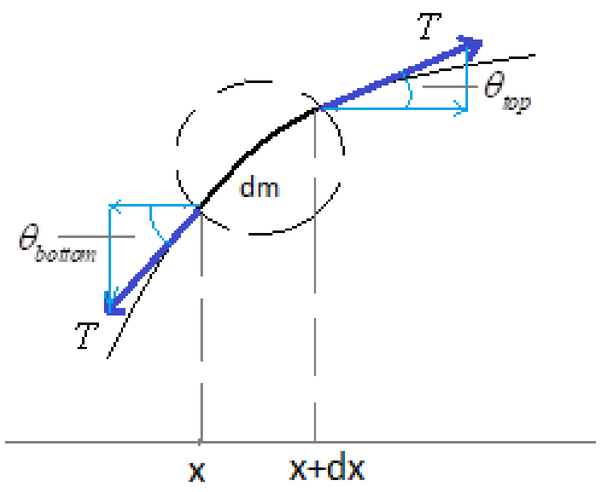
Experiment seems to suggest that a displacement will propagate through the medium keeping same shape, and traveling with a constant velocity, v. In other words D(x,t) = f(x-vt).

But how do we know? And what is v? Gotta use N2L, like we did for springs and pendula. What follows is a derivation which 99% of you will despise, but, you know it's all about the 1%.

B.1 N2L Analysis of Wave Propagation (1D)

Consider for example, a wave propagating along a a string with mass density μ, and under tension T.





$$T_{top(D)} = T\sin\theta_{top} \overset{\text{small angle approximation}}{\approx} T\tan\theta_{top} = TD'(x+dx)$$

$$T_{bottom(D)} = T\sin\theta_{bottom} \overset{\text{small angle approximation}}{\approx} T\tan\theta_{top} = TD'(x)$$

$$\sum \mathbf{F} = m\mathbf{a}$$

$$T\sin\theta_{top} - T\sin\theta_{bottom} = (dm)\frac{d^2D}{dt^2}$$

$$TD'(x+dx) - TD'(x) = (\mu dx)\frac{d^2D}{dt^2}$$

$$T\frac{D'(x+dx) - D'(x)}{dx} = \mu\frac{d^2D}{dt^2}$$

$$T\frac{d^2D}{dx^2} = \mu\frac{d^2D}{dt^2}$$ wave equation for string

B.1 Solution of Wave Equation (1D)

So we have proposed that the solution to this equation is D(x,t) = f(x-vt), where f(x) is the initial displacement of the wave. To find out if this is true, and to determine the velocity of the wave, we must plug this guess into the wave equation:

$$T\frac{d^2D}{dx^2} = \mu\frac{d^2D}{dt^2}$$

$$T\frac{d^2}{dx^2}f(x-vt) = \mu\frac{d^2}{dt^2}f(x-vt)$$

$$T\frac{d}{dx}f'(x-vt)(1) = \mu\frac{d}{dt}f'(x-vt)(-v)$$

$$Tf''(x-vt)(1)(1) = \mu f''(x-vt)(-v)(-v)$$

$$T(1) = \mu(v^2)$$

$$v = \sqrt{T/\mu}$$

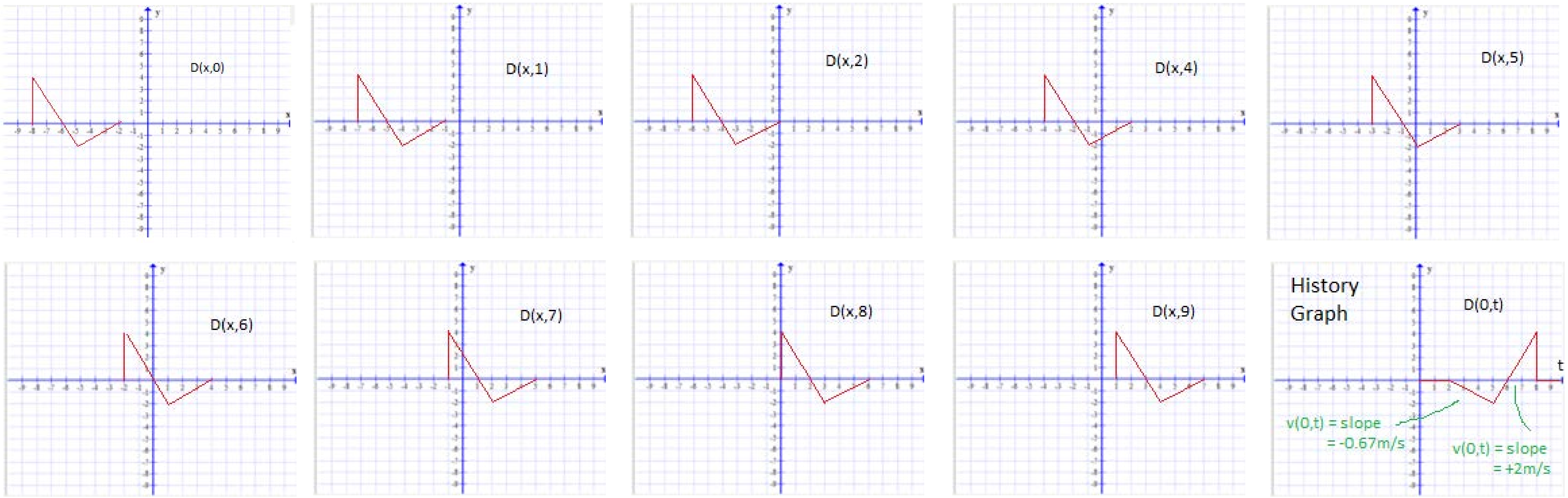
So the fact that f''(x-vt) cancels out on both sides, indicates that it is a valid solution. And moreover we find the bottom formula for the velocity of the wave. So we have, for strings at least:

if	$D(x,0) = f(x)$	
then	$D(x,t) = f(x-vt)$	$v = \sqrt{\dfrac{T}{\mu}}$

The velocity formula should make sense: higher tension → higher velocity
heavier string → smaller velocity

B.1 Determining motion of medium from D(x,0) graph (1D)

Consider an initial wave pulse D(x,0) shown below, which propagates down the string with velocity v = 1m/s. What would the string look like in 1s intervals thereafter (these are ‘snapshot’ graphs)? What would the displacement of a fixed point on the string (say x = 0m in this example) be as a function of time (this is called a ‘history graph’)?



B.1 Determining motion of medium from D(x,0) formula (1D)

Consider an initial wave pulse along a string given by formula:
and propagates down the string with velocity v = 5m/s to the left

$$D(x,0) = \frac{1}{x^2+4}$$

What would be the displacement profile (snapshot function) of the entire string as a function of time?

$$D(x,t) = \frac{1}{(x-vt)^2+4} = \frac{1}{(x-(-5)t)^2+4} = \frac{1}{(x+5t)^2+4}$$

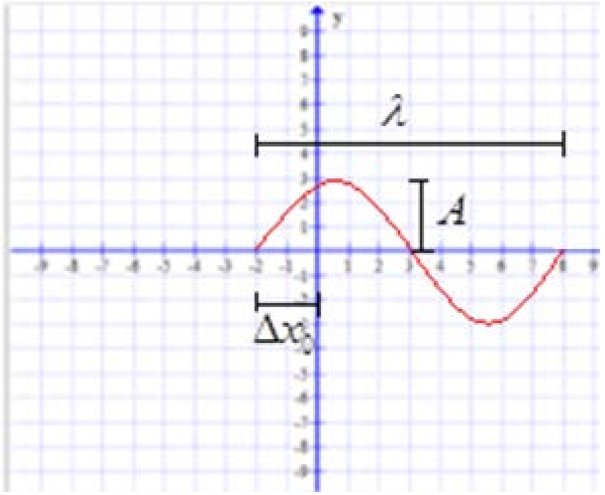
And what would be the displacement and velocity of the point, say, x = -3m, as a function of time (history function)?

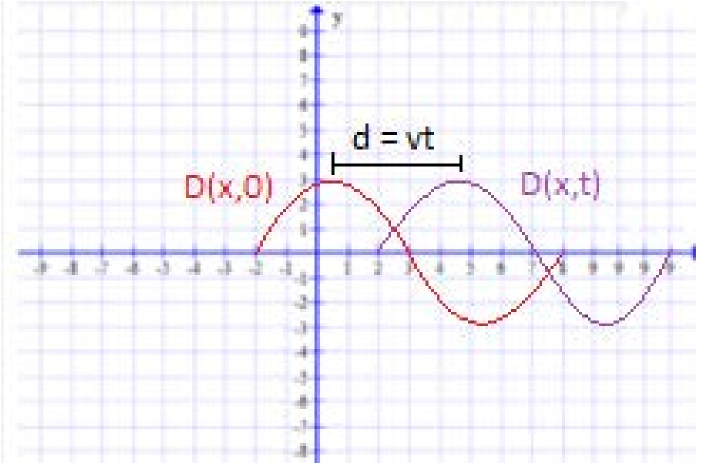
$$D(-3,t) = \frac{1}{(-3+5t)^2+4} = \frac{1}{(3-5t)^2+4}$$

$$v(-3,t) = \frac{dD(-3,t)}{dt} = -\frac{1}{\left[(3-5t)^2+4\right]^2}\cdot 2(3-5t)\cdot(-5) = \frac{10(3-5t)}{\left[(3-5t)^2+4\right]^2}$$

B.1 Sinusoidal Waves (1D)

Most waves produced in nature are sinusoidal, so they bear a little investigation.





$$D(x,0) = f(x) = A\sin(kx + \varphi_0)$$

$$D(x,t) = f(x - vt) = A\sin\left[k(x - vt) + \varphi_0\right] = A\sin\left[kx - \omega t + \varphi_0\right]$$

$A = \text{amplitude}$

$k = \text{'curvature'} = \dfrac{2\pi}{\lambda}$ $\qquad \lambda = \text{wavelength}$

$\varphi_0 = \text{'phase constant'} = \dfrac{2\pi}{\lambda}\Delta x_0$

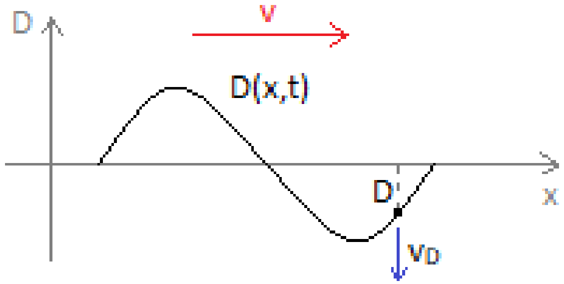
$T = \text{'period'} = \text{time it takes for a wavelength to pass by}$

$f = \text{'frequency'} = \text{rate at which wavelengths pass by} = \dfrac{1}{T}$ [units = Hertz (Hz)]

$\omega = \text{'angular frequency'} = \dfrac{2\pi}{T}$ [units = rad/s]

$v = \dfrac{\lambda}{T} = \lambda f = \omega / k$

B.1 Energy Carried by Sinusoidal Waves (1D)



As a wave passes through a medium, it excites the resident particles into Simple harmonic motion. As such, they will possess kinetic and elastic 'spring' potential energy. We'd like to investigate how much energy will reside in the particles therefore, and also, how quickly the wave will impart this energy to them.

$$E_{particle} = \frac{1}{2} m_{particle} v_D^2 + \frac{1}{2} k D^2 \qquad k = \text{'spring constant' of the string}$$

$$= \frac{1}{2} m_{particle} v_D^2 + \frac{1}{2}\left(m_{particle}\omega^2\right) D^2 \qquad \text{using } \omega = \sqrt{\frac{k}{m_{particle}}}$$

$$= \frac{1}{2} m_{particle}\left[-A\omega\cos(kx-\omega t+\varphi_0)\right]^2 + \frac{1}{2} m_{particle}\omega^2\left[A\sin(kx-\omega t+\varphi_0)\right]^2$$

$$= \frac{1}{2} m_{particle} A^2 \omega^2$$

$$E_{length,dx} = \frac{1}{2}(\mu dx) A^2 \omega^2 \qquad dm = \mu dx = \text{mass in length } dx$$

$$= \frac{1}{2}\mu (A\omega)^2 dx$$

$$E_{length,l} = \int u dx$$

$$u = \text{'energy density'} = \frac{1}{2}\mu (A\omega)^2 \quad [\text{units} = \text{J/m}]$$

$$I = \text{'intensity'} = uv \qquad [\text{units} = \text{J/s} = \text{W}]$$

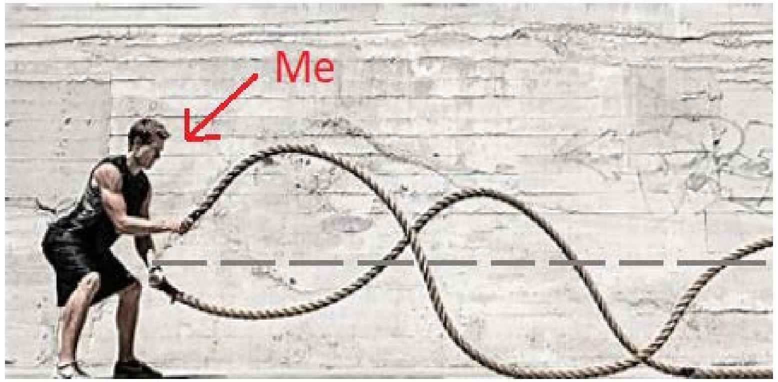
$$P = \text{'power'} = \frac{\text{energy passing past point in time } \Delta t}{\Delta t}$$

$$= \frac{u(v\Delta t)}{\Delta t} = uv = I$$

For example….

Suppose you hulk out on one of those large ropes at the gym.

- It has mass m = 15kg, and length 5m.
- You pull on the rope with force 90N, and shake your arms up and down 1m, twice per second.



1. Write an equation for the bottom rope's displacement D(x,t)

$$A = ? \qquad 0.5\text{m}$$

$$\varphi_0 = \frac{2\pi}{\lambda}\Delta x_0 = ? \qquad \frac{2\pi}{\lambda}\left(-\frac{\lambda}{2}\right) = -\pi\ \text{rad}$$

$$v = \sqrt{\frac{T}{\mu}} = ? \qquad \sqrt{\frac{90\text{N}}{15\,\text{kg}/5\text{m}}} = 5.5\text{m/s}$$

$$k = \frac{2\pi}{\lambda} = ? \qquad \frac{2\pi}{v/f} = \frac{2\pi}{5.5/2} = \frac{2\pi}{2.75} = 2.3\text{m}^{-1}$$

$$\longrightarrow$$

$$\begin{aligned} D(x,t) &= A\sin\left[k(x-vt)+\varphi_0\right] \\ &= 0.5\text{m}\sin\left[2.3(x-5.5t)-\pi\right] \\ &= 0.5\text{m}\sin\left[2.3x-12.7t-\pi\right] \end{aligned}$$

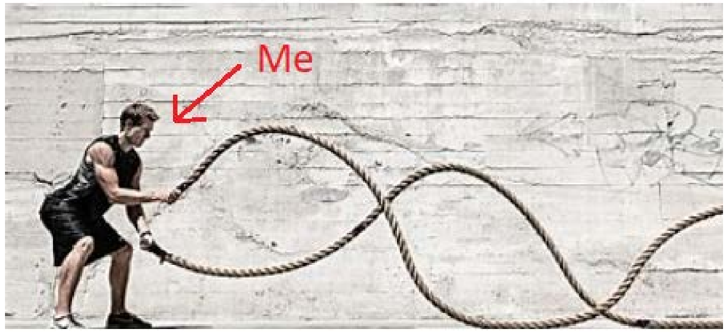
2. What is the displacement, velocity, and acceleration of the point x = 1.5m down the rope, at time t = 3s?

$$D(x,t) = 0.5\sin\left[2.3x-12.7t-\pi\right] \quad\rightarrow\quad D(1.5,3) = -0.046\text{m}$$

$$v(x,t) = \frac{d}{dt}D(x,t) = (-12.7)\cdot(0.5)\cos\left[2.3x-12.7t-\pi\right] \quad\rightarrow\quad v(1.5,3) = -6.3\,\text{m/s}$$

$$a(x,t) = \frac{d^2}{dt^2}D(x,t) = (-12.7)^2\cdot -(0.5)\sin\left[2.3x-12.7t-\pi\right] \quad\rightarrow\quad a(1.5,3) = 7.45\ \text{m/s}^2$$

Not done yet…..



3. What are the maximum values of D, v, a for that (or any) point?

$$D(x,t) = 0.5\sin\left[2.3x - 12.7t - \pi\right]$$

$$v(x,t) = \frac{dD}{dt} = (-12.7)\cdot(0.5)\cos\left[2.3x - 12.7t - \pi\right]$$

$$a(x,t) = \frac{d^2D}{dt^2} = (-12.7)^2\cdot -(0.5)\sin\left[2.3x - 12.7t - \pi\right]$$

$$\rightarrow D_{\max} = 0.5\text{m}$$

$$\rightarrow v_{\max} = (12.7)(0.5) = 6.3\text{m/s}$$

$$\rightarrow a_{\max} = (12.7)^2(0.5) = 80.6\ \text{m/s}^2$$

4. How much energy resides in each wavelength?

$$E = ul$$

$$= \frac{1}{2}\mu(A\omega)^2 l$$

$$\mu = \frac{m}{l} = \frac{15\,\text{kg}}{5\text{m}} = 3\,\text{kg/ m},\qquad \omega = 2\pi f = 2\pi(2) = 4\pi\text{rad/s},\qquad l = \lambda = \frac{v}{f} = \frac{5.5\text{m/s}}{2\text{Hz}} = 2.75\text{m}$$

$$= \frac{1}{2}\left(3\frac{\text{kg}}{\text{m}}\right)\left[(0.5\text{m})(4\pi\ \text{rad/s})\right]^2(2.75\text{m})$$

$$= \left(59\frac{\text{J}}{\text{m}}\right)(2.75\text{m}) = 163\,\text{J}$$

5. What is the intensity of this wave you're creating? What is your power output?

$$I = uv = (59\,\text{J/m})(5.5\,\text{m/s}) = 324\,\text{W}$$

$$P = I = 324\,\text{W}$$